Lecturered By Dr. H. K. Yadav.
Deptt. of Mathematics, Mandari College, JBLr
B.Sc-part-I(H), paper-I, Date - 01-06-2020.
01-06-2020

Q.1. State and prove Cayley's theorem:

Soln: Statement: → Every finite group G is isomorphic to its permutation Group.

proof: Let G be a finite group of order n such that $G = \{a_1, a_2, a_3, \ldots a_n\}$.

Let $a \in G$. Define a map $f_a : G \Rightarrow G$ given by

$f_a(x) = ax, \ \forall \ x \in G$ ——— (I)

f_a is one-one.

Let $f_a(x_1) = f_a(x_2), \ \forall x_1, x_2 \in G$

$\Rightarrow \ ax_1 = ax_2 \Rightarrow x_1 = x_2$

f_a is onto.

$f_a : G \to G$ is one-one and G is finite

f_a is one-one map of a finite set G onto itself.

$\Rightarrow f_a$ is onto. Thus, f_a is one-one map of a finite set G onto itself. It means that f_a is a permutation of degree n. Here

$$f_a = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ aa_1 & aa_2 & \cdots & aa_n \end{pmatrix}$$

The elements $aa_1, aa_2, \cdots aa_n$ are all distinct element of G.

We write, $G' = \{f_a : a \in G\}$, then G' is a set of permutations of degree n.

Now, we claim that $(G', *)$ is a group, where $(*)$ denotes permutations multiplication.

Let $a, b, c \in G$ be arbitrary and e be the identity in G.

Let a^{-1} denote inverse of a in G.

So, that $a^{-1}a = aa^{-1} = e$.

(I) Closure property: $f_a f_b \in G' \Rightarrow f_a f_b \in G'$

Let us consider $(f_a f_b)(x) = f_a[f_b(x)] = f_a(bx)$

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$= a(bx) = (ab)x$ [by associativity in G]

$= f_{ab}(x)$

$\Rightarrow f_a f_b = f_{ab}$ ——— II

$a, b \in G \Rightarrow ab \in G \Rightarrow f_{ab} \in G' \Rightarrow f_a f_b \in G'$

(II) Associativity: Let $a, b, c \in G$

$\Rightarrow (ab)c = a(bc) \Rightarrow f_{(ab)c} = f_{a(bc)}$

$\Rightarrow f_{(ab)} f_c = f_a f_{bc} \Rightarrow (f_a f_b) f_c = f_a (f_b f_c)$

III Existence of identity:

$a, e \in G \Rightarrow f_e \in G'$ and $ae = ea = a$

$\Rightarrow f_{ae} = f_{ea} = f_a \Rightarrow f_a f_e = f_e f_a = f_a$

$\Rightarrow f_e \in G'$ is identity element of G'.

IV Existence of inverse: $a \in G \Rightarrow a \cdot a^{-1} \in G \Rightarrow f_a \cdot f_{a^{-1}} \in G'$

Also, $a a^{-1} = a^{-1} a = e$

$f_{aa^{-1}} = f_{(a^{-1}a)} = f_e$ or, $f_a f_{a^{-1}} = f_{a^{-1}} f_a = f_e$

$\Rightarrow f_{a^{-1}} \in G'$ is the inverse of $f_a \in G$.

(V) Order of G':

For $O(G) = n$, $G' = \{f_a : a \in G\} \Rightarrow O(G') = n$

Therefore, we have $(G', *)$ is a finite group of order n.

Now, we claim that $(G, *) \cong (G', *)$

Now, we define a map $g : G \to G'$ such that $g(x) = f_x, \forall x \in G$.

(I) g is one-one:

For $g(x_1) = g(x_2)$, $\forall x_1, x_2 \in G$

$\Rightarrow f_{x_1} = f_{x_2} \Rightarrow f_{x_1}(x) = f_{x_2}(x); \forall x \in G$.

$\Rightarrow x_1 x = x_2 x \Rightarrow x_1 = x_2$ [by. I]

(II) g is onto:

For any $f_a \in G' \Rightarrow a \in G$ such that $g(a) = f_a$

(III) g preserves composition in G and G':

For $g(x_1 x_2) = f_{x_1 x_2}$ [where $x_1, x_2 \in G \Rightarrow x_1 x_2 \in G$]

$= f_{x_1} f_{x_2} = g(x_1) \cdot g(x_2)$

Hence, g is an isomorphism of G onto G' and hence

$G \cong G'$.

Proved.